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ORTHOGONAL POLYNOMIALS AND HYPERGROUPS II—THE SYMMETRIC CASE

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ABSTRACT. The close relationship between orthogonal polynomial sequences and polynomial hypergroups is further studied in the case of even weight function, cf. [18]. Sufficient criteria for the recurrence relation of orthogonal polynomials are given such that a polynomial hypergroup structure is determined on $\mathbb{N}_0$. If the recurrence coefficients are convergent the dual spaces are determined explicitly. The polynomial hypergroup structure is revealed and investigated for associated ultraspherical polynomials, Pollaczek polynomials, associated Pollaczek polynomials, orthogonal polynomials with constant monic recursion formula and random walk polynomials.

1. POLYNOMIAL HYPERGROUPS

In [18] we demonstrated a close relationship between certain hypergroups on $\mathbb{N}_0$ and certain orthogonal polynomial sequences. In this paper we discuss other basic properties concerning these hypergroups—we call them *polynomial hypergroups*—and give many examples based on certain classes of orthogonal polynomial sequences. Our main reference is [18]. Notation and results from [18] are used throughout.

The best known class corresponding to a polynomial hypergroup are the Jacobi polynomials $(P_n^{(\alpha,\beta)}(x))_{n=0}^{\infty}$ for $\alpha \geq \beta > -1$, $\alpha+\beta+1 \geq 0$ (see [18, 3(a)]). The basic property (P) holds as is shown by Gasper [11]. In [12] Gasper also proved that $(P_n^{(\alpha,\beta)}(x))_{n=0}^{\infty}$ defines a dual hypergroup structure on the interval $[-1, 1]$. Recently, Connett and Schwartz [9] proved that the Jacobi polynomials with $\alpha \geq \beta > -1$ and either $\beta \geq -1/2$ or $\alpha+\beta \geq 0$ are the only ones which define a dual hypergroup structure on $[-1, 1]$. However, contrary to $[-1, 1]$ on the discrete space $\mathbb{N}_0$, there exists a variety of hypergroups different in structure from Jacobi polynomials. Hypergroups were introduced independently by Dunkl [10], Jewett [16] and Spector [27] in the early seventies. Here we will use [16] as reference.

There are many applications where the hypergroup structure of $\mathbb{N}_0$ can be exploited (see e.g. [20–23, 32–34]). We reveal further polynomial hypergroups and determine their recurrence relations, their Haar weights and specific properties of the corresponding polynomials. In the following we consider the case of *even* orthogonalization measures only. Also, we choose $x = 1$ as the point

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which determines the unit character, i.e., we have to determine the normalization of the orthogonal polynomials such that they have the value 1 at $x = 1$.

Given an orthogonal polynomial sequence $(\widetilde{P}_n(x))_{n=0}^{\infty}$ with a known recurrence relation

$$(1.1)\qquad x\widetilde{P}_n(x) = e_n\widetilde{P}_{n+1}(x) + d_n\widetilde{P}_{n-1}(x), \qquad n = 1, 2, \ldots,$$

and

$$(1.2)\qquad \widetilde{P}_0(x) = 1, \qquad \widetilde{P}_1(x) = x/e_0,$$

we have to take two steps in order to reveal the corresponding hypergroup structure, if there is any.

First, after renormalizing $P_n(x) = k_n\widetilde{P}_n(x)$ with $P_n(1) = 1$ in the related recurrence

$$(1.3)\qquad xP_n(x) = (1 - c_n)P_{n+1}(x) + c_nP_{n-1}(x), \qquad n = 1, 2, \ldots,$$

and

$$(1.4)\qquad P_0(x) = 1, \qquad P_1(x) = x$$

we have to show that condition $0 < c_n < 1$ is satisfied for all $n = 1, 2, \ldots$. Polynomials generated by (1.3), (1.4) are sometimes called random walk polynomials, see [17].

Second, we have to check whether property (P) is fulfilled. For this problem there is, in addition to a known criterion by Askey (see Corollary 1.4 below), a recent result of Szwarc.

A normalization, often used, is setting the leading coefficient to 1. These polynomials $\phi_n(x)$ are called monic and satisfy (if the orthogonalization measure is even)

$$(1.5)\qquad x\phi_n(x) = \phi_{n+1}(x) + \gamma_n\phi_{n-1}(x), \qquad n = 1, 2, \ldots,$$

and, of course,

$$(1.6)\qquad \phi_0(x) = 1, \qquad \phi_1(x) = x.$$

The first step in our program may be characterized by means of chain sequences. For completeness' sake, we include the definition of a chain sequence and refer to [6, p. 91 ff.] for basic facts.

Definition. A sequence $(\gamma_n)_{n=1}^{\infty}$ is a chain sequence, if there is a sequence $(d_n)_{n=0}^{\infty}$ such that $0 \le d_0 < 1$, $0 < d_n < 1$, and $\gamma_n = d_n(1 - d_{n-1})$ for $n = 1, 2, \ldots$. $(d_n)_{n=0}^{\infty}$ is called parameter sequence for $(\gamma_n)_{n=1}^{\infty}$. A parameter sequence $(c_n)_{n=0}^{\infty}$ is called minimal, if $c_0 = 0$.

The following observation is well known:

Proposition (1.1). *The orthogonal polynomial sequence $(\widetilde{P}_n(x))_{n=0}^{\infty}$ with the recurrence relation (1.1), (1.2) can be normalized to $P_n(x) = k_n\widetilde{P}_n(x)$ such that (1.3), (1.4) hold with $0 < c_n < 1$ for $n = 1, 2, \ldots$, if and only if $(\gamma_n)_{n=1}^{\infty}$ is a chain sequence, where*

$$(1.7)\qquad \gamma_n = d_ne_{n-1}, \qquad n = 1, 2, \ldots.$$

The sequence $(c_n)_{n=0}^{\infty}$ (with $c_0 = 0$) is the minimal parameter sequence for $(\gamma_n)_{n=1}^{\infty}$. Moreover, k_n is given by

$$k_n = \prod_{k=0}^{n-1} \frac{e_k}{1 - c_k} \quad \text{for } k = 2, 3, \ldots. \tag{1.8}$$

In [7] Chihara characterizes minimal parameter sequences for a given chain sequence in a very simple and interesting manner. A way to determine the c_n in the recurrence relation (1.3), (1.4) as functions of the e_n, d_n resp. γ_n is to compute $\widetilde{P}_n(1)$ resp. $\phi_n(1)$. In fact, since

$$P_n(x) = \frac{1}{\widetilde{P}_n(1)} \widetilde{P}_n(x) = \frac{1}{\tilde{\phi}_n(1)} \phi_n(x),$$

we obtain

$$c_n = \frac{\widetilde{P}_{n-1}(1)}{\widetilde{P}_n(1)} d_n = \frac{\phi_{n-1}(1)}{\phi_n(1)} \gamma_n \quad \text{for } n = 0, 1, 2, \ldots \tag{1.9}$$

(provided $\widetilde{P}_n(1)$ resp. $\phi_n(1)$ is not zero, and that we set $\widetilde{P}_{-1}(x) = \phi_{-1}(x) = 0$).

Further, by (1.1) resp. (1.5) we have

$$1 - c_n = \frac{\widetilde{P}_{n+1}(1)}{\widetilde{P}_n(1)} e_n = \frac{\phi_{n+1}(1)}{\phi_n(1)}. \tag{1.10}$$

The following result is rather useful and has a straightforward proof.

Proposition (1.2). *Let $(\widetilde{P}_n(x))_{n=0}^{\infty}$ be defined recursively by (1.1), (1.2) and assume that $e_n = 1 - d_n$, $d_n \neq 1$ for each $n = 0, 1, 2, \ldots$. Then*

$$\widetilde{P}_n(1) = 1 + \sum_{k=1}^{n} \frac{d_0 \cdots d_{k-1}}{(1 - d_0) \cdots (1 - d_{k-1})} \tag{1.11}$$

holds for $n = 1, 2, \ldots$.

If the recurrence coefficients in (1.1), (1.2) can be written as

$$d_n = \frac{\lambda_n}{\lambda_n + \mu_{n+1}} \quad \text{and} \quad e_n = \frac{\mu_{n+1}}{\lambda_n + \mu_{n+1}}, \qquad n = 0, 1, 2, \ldots, \tag{1.12}$$

where $(\lambda_n)_{n=0}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ are two sequences of positive numbers, up to λ_0, for which we only assume that $\lambda_0 \neq -\mu_1$, then Proposition (1.2) may be used as follows

$$\widetilde{P}_n(1) = 1 + \sum_{k=1}^{n} \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}, \qquad n = 1, 2, \ldots. \tag{1.13}$$

A result of Szwarc [29] will be useful now to establish our second step, namely Theorem 1.3 below.

Theorem (1.3). *Assume an orthogonal polynomial sequence $(\widetilde{P}_n(x))_{n=0}^{\infty}$ is defined by (1.1), (1.2) such that*

(a) *$(d_n e_{n-1})_{n=1}^{\infty}$ is a chain sequence,*

(b) *$(d_n)_{n=1}^{\infty}$ and $(e_n)_{n=1}^{\infty}$ are sequences of positive numbers with $d_n \leq e_n$, $d_n \leq d_{n+1}$ and $d_n + e_n \leq d_{n+1} + e_{n+1}$ for $n = 1, 2, \ldots$ and $e_0 \leq d_1 + e_1$.*

Then the minimal parameter sequence $(c_n)_{n=0}^{\infty}$ *of* $(d_n e_{n-1})_{n=1}^{\infty}$ *determines a polynomial hypergroup structure on* $\mathbb{N}_0$.

Proof. By assumption (a) we have a sequence of numbers $c_n \in]0, 1[$, $n = 1, 2, \ldots$ such that (1.3), (1.4) holds, compare Proposition (1.1). By (1.9) we conclude from the positivity of c_n and d_n by induction that each $\widetilde{P}_n(1)$ is positive. By [29, Theorem 1]

$$\widetilde{P}_m(x)\widetilde{P}_n(x) = \sum_{k=|m-n|}^{m+n} \tilde{g}(m, n, k)\widetilde{P}_k(x)$$

with $\tilde{g}(m, n, k) \geq 0$ for $m, n = 1, 2, \ldots$. Since $P_n(x) = \widetilde{P}_n(x)/\widetilde{P}_n(1)$ we have

$$P_m(x)P_n(x) = \sum_{k=|m-n|}^{m+n} g(m, n, k)P_k(x)$$

with

$$g(m, n, k) = \tilde{g}(m, n, k)\frac{\widetilde{P}_k(1)}{\widetilde{P}_m(1)\widetilde{P}_n(1)} \geq 0. \quad \square$$

Corollary (1.4) (Askey). *Assume a monic orthogonal polynomial sequence* $(\phi_n(x))_{n=0}^{\infty}$ *is defined by* (1.5), (1.6) *such that*

(a) $(\gamma_n)_{n=1}^{\infty}$ *is a chain sequence,*
(b) $(\gamma_n)_{n=1}^{\infty}$ *is a nondecreasing sequence.*

Then the minimal parameter sequence $(c_n)_{n=0}^{\infty}$ *of* $(\gamma_n)_{n=1}^{\infty}$ *determines a polynomial hypergroup structure on* $\mathbb{N}_0$.

Proof. Use Theorem 1.3 with $d_n = \gamma_n$ and $e_n = 1$. $\square$

Corollary (1.5). *Let an orthogonal polynomial sequence* $(\widetilde{P}_n(x))_{n=0}^{\infty}$ *be given by* (1.1), (1.2), *where*

$$d_n = \frac{\lambda_n}{\lambda_n + \mu_{n+1}} \quad \text{and} \quad e_n = \frac{\mu_{n+1}}{\lambda_n + \mu_{n+1}}, \qquad n = 0, 1, 2, \ldots$$

with $\lambda_n, \mu_n > 0$ *for* $n = 1, 2, \ldots$, *and* $\lambda_0 \geq 0$. *Assume*

(a) $(d_n e_{n-1})_{n=1}^{\infty}$ *is a chain sequence,*
(b) $\lambda_n \leq \mu_{n+1}$ *for* $n = 1, 2, \ldots$, *and* $(\mu_{n+1}/\lambda_n)_{n=1}^{\infty}$ *is a nonincreasing sequence.*

Then, the minimal parameter sequence $(c_n)_{n=0}^{\infty}$ *of* $(d_n e_{n-1})_{n=1}^{\infty}$ *determines a polynomial hypergroup structure on* $\mathbb{N}_0$.

Proof. It is easy to verify that the conditions of (b) in Theorem (1.3) are equivalent to the conditions (b) above. $\square$

If

$$d_n = \frac{\lambda_n}{\lambda_n + \mu_{n+1}}, \qquad e_n = \frac{\mu_{n+1}}{\lambda_n + \mu_{n+1}}$$

as above, obviously $(d_n e_{n-1})_{n=1}^{\infty}$ is a chain sequence provided $0 \leq d_0 < 1$, which is equivalent to $\lambda_0 \geq 0$. In order to cover the case $\lambda_0 < 0$ we prove the following result:

Proposition (1.6). *Assume* γ_n *is given as*

$$\gamma_n = \frac{\lambda_n \mu_n}{(\lambda_{n-1} + \mu_n)(\lambda_n + \mu_{n+1})}, \qquad n = 1, 2, \dots,$$

where $\lambda_n, \mu_n > 0$ *for* $n = 1, 2, \dots,$ *and* $\lambda_0 < 0$, $\lambda_0 \neq -\mu_1$. *If*

$$0 < \lambda_n - \lambda_{n-1} \leq \mu_n - \mu_{n-1} \quad \text{for } n = 1, 2, \dots$$

holds (*with* $\mu_0 = 0$), *then* $(\gamma_n)_{n=1}^{\infty}$ *is a chain sequence.*

Proof. We construct the minimal parameter sequence $(c_n)_{n=0}^{\infty}$ recursively by showing

$$0 < c_n \leq \frac{\mu_n}{\lambda_n + \mu_{n+1}} < 1 \quad \text{for } n = 1, 2, \dots, \tag{1.14}$$

where c_{n+1} is defined by the recurrence $c_{n+1} = \gamma_{n+1}/(1-c_n)$. With $c_0 = 0$ and $c_1 = \gamma_1$ we see that

$$0 < c_1 = \frac{\lambda_1 \mu_1}{(\lambda_0 + \mu_1)(\lambda_1 + \mu_2)} \leq \frac{\mu_1}{\lambda_1 + \mu_2} < 1.$$

Assume $c_1, \dots, c_n$ are constructed and satisfy (1.14). Then

$$0 < \frac{\lambda_n + \mu_{n+1} - \mu_n}{\lambda_n + \mu_{n+1}} \leq 1 - c_n < 1.$$

Hence,

$$\begin{aligned} c_{n+1} = \frac{\gamma_{n+1}}{1 - c_n} &= \frac{\lambda_{n+1}\mu_{n+1}}{(\lambda_n + \mu_{n+1})(\lambda_{n+1} + \mu_{n+2})(1 - c_n)} \\ &\leq \frac{\mu_{n+1}\lambda_{n+1}}{(\lambda_{n+1} + \mu_{n+2})(\lambda_n + \mu_{n+1} - \mu_n)} \leq \frac{\mu_{n+1}}{\lambda_{n+1} + \mu_{n+2}}. \qquad \square \end{aligned}$$

The Haar measure plays a particular part in the theory of hypergroups (see [16]). There are various possibilities to calculate the Haar weights $h(n)$ of polynomial hypergroups, cf. [18], defined as

$$h(0) = 1, \qquad h(n) = h(n-1)\frac{1 - c_{n-1}}{c_n} = \frac{(1 - c_0)\cdots(1 - c_{n-1})}{c_1 \cdots c_n}, \tag{1.15}$$

where $(c_n)_{n=0}^{\infty}$ is the minimal parameter sequence for $(\gamma_n)_{n=1}^{\infty}$. With the notation from above by (1.9) and (1.10) we have

$$h(n) = \frac{e_0 \cdots e_{n-1}(\widetilde{P}_n(1))^2}{d_1 \cdots d_n} = \frac{(\phi_n(1))^2}{\gamma_1 \cdots \gamma_n}. \tag{1.16}$$

If d_n and e_n are given as in (1.12), it follows that

$$h(n) = \frac{\mu_1 \cdots \mu_n(\lambda_n + \mu_{n+1})(\widetilde{P}_n(1))^2}{\lambda_1 \cdots \lambda_n(\lambda_0 + \mu_1)}. \tag{1.17}$$

Finally, we consider the orthonormal version $(p_n(x))_{n=0}^{\infty}$. Assuming that the orthonormalization measure π is a probability measure, we have the recurrence relation

$$x p_n(x) = A_{n+1} p_{n+1}(x) + A_n p_{n-1}(x), \qquad n = 1, 2, \dots, \tag{1.18}$$

and

$$p_0(x) = 1, \qquad p_1(x) = \frac{1}{A_1}x \tag{1.19}$$

with $A_n = \sqrt{\gamma_n}$. It is readily deduced that $p_n(x) = \sqrt{h(n)}P_n(x)$, where $P_n(x)$ is normalized as $P_n(1) = 1$. Many asymptotic results originating in various problems of application, i.e., approximation, prediction, refer to orthonormal polynomial sequences. In particular, leading coefficients play an important part, see [25]. The leading coefficient σ_n of $P_n(x)$, i.e., $P_n(x) = \sigma_n x^n + \cdots$, is given by

$$\sigma_n = \prod_{k=0}^{n-1} \frac{1}{1-c_k} = \left(\prod_{k=0}^{n-1} \frac{1}{e_k}\right) / \widetilde{P}_n(1) = \frac{1}{\phi_n(1)}, \qquad n = 1, 2, \ldots, \tag{1.20}$$

whereas the leading coefficient ρ_n of $p_n(x)$, i.e. $p_n(x) = \rho_n x^n + \cdots$, is

$$\begin{aligned} \rho_n &= \prod_{k=1}^{n} \frac{1}{A_k} = \left(\prod_{k=1}^{n} \frac{1}{\gamma_k}\right)^{1/2} = \left(\prod_{k=1}^{n} \frac{1}{d_k e_{k-1}}\right)^{1/2} \\ &= \left(\prod_{k=1}^{n} \frac{1}{(1-c_{k-1})c_k}\right)^{1/2} = (h(n))^{1/2}\sigma_n. \end{aligned} \tag{1.21}$$

2. Dual spaces and other properties

The dual space of polynomial hypergroups can be identified with a compact subset of $\mathbb{R}$. Explicitely, the space $\widehat{\mathbb{N}}_0$ of all hermitian characters of the hypergroup $\mathbb{N}_0$ is homeomorphic to

$$D_s = \{x \in \mathbb{R}: |P_n(x)| \leq 1 \text{ for all } n \in \mathbb{N}_0\}$$

and the space of all (not necessarily hermitian) characters to

$$D = \{z \in \mathbb{C}: |P_n(z)| \leq 1 \text{ for all } n \in \mathbb{N}_0\}$$

(see [18, Proposition 4]). We are in a position to describe the character spaces easily when $(c_n)_{n=0}^{\infty}$ is convergent, and, where the orthogonalization measure is even, as is assumed throughout. Parts of the following theorem are known (see [30 and 35]). To begin with we show

Lemma (2.1). *Let $(p_n)_{n=0}^{\infty}$ be an orthonormal polynomial sequence with respect to π. If $x \in \operatorname{supp}\pi$ and $|p_n(x)| \geq c > 0$ for all $n \in \mathbb{N}_0$, then x is not an isolated point of* $\operatorname{supp}\pi$.

Proof. Assume x is an isolated point of $\operatorname{supp}\pi$. Then, there exists a continuous function f with compact support, such that $f(x) = 1$ and $f|\operatorname{supp}\pi\backslash\{x\} = 0$. Hence, $\langle f, p_n\rangle = \int_{\mathbb{R}} f(x)p_n(x)\,d\pi(x) = \pi(\{x\})p_n(x)$. Since $f \in L^2(\pi)$, we have $\langle f, p_n\rangle \to 0$ as $n \to \infty$, a contradiction. □

Theorem (2.2). *Assume that $(P_n)_{n=0}^{\infty}$, given by* (1.3), (1.4), *determines a polynomial hypergroup on $\mathbb{N}_0$. Further, assume $c_n \to c$ as $n \to \infty$, and let π denote the corresponding orthogonalization measure.*

(i) *The only possible values of c are $0 < c \leq \frac{1}{2}$.*

(ii) *If* $0 < c \le \frac{1}{2}$, *then*

$$\operatorname{supp}\pi = [-2\sqrt{c(1-c)}, 2\sqrt{c(1-c)}], \qquad D_s = [-1, 1],$$

and

$$D = \{z \in \mathbb{C}: |z - 2\sqrt{c(1-c)}| + |z + 2\sqrt{c(1-c)}| \le 2\}.$$

Proof. (1) Note that $c > \frac{1}{2}$ is not possible. In fact, $c > \frac{1}{2}$ would imply that the c_n are finally greater than $\frac{1}{2}$, which contradicts $h(n) = g(n, n, 0)^{-1} > 1$.

(2) Consider the case $0 < c \le \frac{1}{2}$. In the notation of P. Nevai [25] we have $\pi \in M(0, b)$, where $b = 2\sqrt{c(1-c)}$. A theorem of Blumenthal (see [25, Theorem 7, p. 23]) gives $\operatorname{supp}\pi = [-b, b] \cup S$, S a countable subset of $\mathbb{R}$ whose derived set is contained in $\{-b, b\}$. We show that $S \subseteq [-b, b]$. If S is not a subset of this interval, $\alpha_0 = \max(\operatorname{supp}\pi) > 0$ is an isolated point of $\operatorname{supp}\pi$. Remember the symmetry of $\operatorname{supp}\pi$. Then, by the separating property of zeros in $P_n(x)$ (see [6]), we have $P_n(\alpha_0) > 0$ for all $n \in \mathbb{N}$. We set $Q_n(x) = P_n(\alpha_0 x)/P_n(\alpha_0)$ and easily demonstrate that $(Q_n(x))_{n=0}^{\infty}$ determines a polynomial hypergroup. The support of the corresponding orthogonalization measure contains 1 as an isolated point. The corresponding orthonormal polynomials $q_n(x)$ satisfy $q_n(1) = \sqrt{h_Q(n)} \ge 1$, where $h_Q(n)$ denotes the Haar weights of $(Q_n(x))_{n=0}^{\infty}$. But this is contradicting Lemma (2.1):

(3) Now we prove $D = \{z \in \mathbb{C}: |z - b| + |z + b| \le 2\}$, which implies $D_s = [-1, 1]$, too. Bear in mind $E = \{z \in \mathbb{C}: |z - b| + |z + b| \le 2\}$ is an ellipse with the focus points $-b$ and b and with the boundary $\{(\cos t, (1-2c)\sin t): t \in [0, 2\pi[\}$. The image of $\{u \in \mathbb{C}: 0 < |u| < 1/(1-c)\}$ under the Joukovsky function $\phi(u) = 1/u + c(1-c)u$ is $\mathbb{C}\backslash E$, whereas E is the image of $\{u \in \mathbb{C}: 1/(1-c) \le |u| \le 2/b\}$. The circle $\{u \in \mathbb{C}: |u| = 2/b\}$ is mapped onto the interval $[-b, b]$, the circle $\{u \in \mathbb{C}: |u| = 1/(1-c)\}$ onto the boundary of E. Further, by Poincaré's theorem (see [25, Theorem 13, p. 33]) we have for $z \in \mathbb{C}\backslash[-b, b]$,

$$\frac{z}{b} + \sqrt{\left(\frac{z}{b}\right)^2 - 1} = \lim_{n\to\infty} \frac{p_n(z)}{p_{n-1}(z)} = \lim_{n\to\infty} \frac{\sqrt{h(n)}P_n(z)}{\sqrt{h(n-1)}P_{n-1}(z)}.$$

Hence,

$$\lim_{n\to\infty} \frac{P_n(z)}{P_{n-1}(z)} = \frac{1}{2(1-c)}(z + \sqrt{z^2 - b^2}).$$

(We take the branch of the square root above such that $|z + \sqrt{z^2 - b^2}| > 1$, whenever $z \in \mathbb{C}\backslash[-b, b]$.)

For any $z \in \mathbb{C}\backslash[-b, b]$ there is an $u \in \mathbb{C}$ with $|u| < 2/b$ such that $\phi(u) = z$. An easy calculation gives

$$\frac{1}{2(1-c)}\left(\phi(u) + \sqrt{\phi(u)^2 - b^2}\right) = \frac{1}{u(1-c)}.$$

For $z \in \mathbb{C}\backslash E$ we have $\phi(u) = z$ with $0 < |u| < 1/(1-c)$. Therefore,

$$\lim_{n\to\infty} \frac{|P_n(z)|}{|P_{n-1}(z)|} = \frac{1}{|u|(1-c)} > 1,$$

and z cannot be an element of D, whenever $z \in \mathbb{C}\backslash E$. We remark that statement (ii) is now proved completely for the case $c = \frac{1}{2}$. In fact, for $c = \frac{1}{2}$ the results above and [18, Proposition 4] yield

$$[-1, 1] = \operatorname{supp} \pi \subseteq D_s \subseteq [-1, 1],$$

implying also $D = D_s$ since $D \subseteq \mathbb{R}$. We continue our proof for $0 < c < \frac{1}{2}$. Given $z \in \mathring{E}\backslash[-b, b]$ we find a u so that $1/(1-c) < |u| < 2/b$ and $\phi(u) = z$ ($\mathring{E}$ being the ellipse without its boundary). Hence,

$$\lim_{n\to\infty} \frac{|P_n(z)|}{|P_{n-1}(z)|} = \frac{1}{|u|(1-c)} < 1,$$

and z is an element of D, whenever $z \in \mathring{E}\backslash[-b, b]$. Finally, as D is a closed set (see [18, Proposition 4]) we obtain $D = E$.

(4) Remains to prove that $c = 0$ is not possible. Applying the arguments used in (2) we get $\operatorname{supp} \pi = \{0\}$, which is a contradiction to $|\operatorname{supp} \pi| = \infty$. □

An important asymptotic property we called condition (H) is used in [23 and 24]. An orthogonal polynomial sequence $(\widetilde{P}_n)_{n=0}^\infty$ satisfies (H), if

$$\text{(H)} \qquad \lim_{n\to\infty} \frac{h(n)}{\sum_{k=0}^n h(k)} = 0.$$

Two recent results give sufficient criteria for (H).

Proposition (2.3). *Assume* $(\gamma_n)_{n=1}^\infty$ *is a chain sequence with minimal parameter sequence* $(c_n)_{n=0}^\infty$. *Then* (H) *is satisfied, if one of the following conditions is valid:*

(i) $\gamma_n \to \frac{1}{4}$, *when* $n \to \infty$,
(ii) $\frac{c_n}{1-c_{n-1}} \to 1$, *when* $n \to \infty$.

Furthermore, (H) *does not hold if* $c_n \to c$, $0 < c < \frac{1}{2}$.

Proof. Note first that $h(n) > 0$ holds. Now (ii) is proved in [24, Proposition 1] and (i) is contained in [37, Theorem (2.1)]. In [37] substantially deeper results are shown than those used here. If $c_n \to c$ with $0 < c < \frac{1}{2}$ we have

$$\lim_{n\to\infty} \frac{h(n+1)}{h(n)} = \frac{1-c}{c} > 1.$$

Lemma (3.3) of [37] yields

$$\lim_{n\to\infty} \frac{h(n)}{\sum_{k=0}^n h(k)} = \frac{1-2c}{1-c}. \quad \square$$

Remark. If $(\gamma_n)_{n=1}^\infty$ is a chain sequence, condition (i) of Proposition (2.3) is equivalent to $c_n \to \frac{1}{2}$. This is a special case of the more general result in Chihara [6, Chapter 3, Theorem 6.4].

A property of orthogonal polynomials $(P_n)_{n=0}^\infty$ often used by Voit [31–34] is condition (T): Let $(T_n)_{n=0}^\infty$ be the Tchebichef polynomials of the first kind which are defined by $T_n(x) = \cos(n \arccos x)$, $x \in [-1, 1]$. The connection coefficients $a_{n,k}$, $n \in \mathbb{N}_0$, $k = 0, \ldots, n$, are uniquely determined by

$$P_n(x) = \sum_{k=0}^n a_{n,k} T_k(x).$$

$(P_n)_{n=0}^\infty$ satisfies (T), if all $a_{n,k}$ are nonnegative. (Note that Voit uses a slightly different definition of property (T).) Since the recurrence coefficients β_n of the monic version of the $T_n(x)$ are $\beta_1 = \frac{1}{2}$, $\beta_n = \frac{1}{4}$, $n = 2, 3, \ldots$, Theorem 1 of Askey's paper [1], gives

Proposition (2.4). *Let $(P_n)_{n=0}^\infty$ be an orthogonal polynomial sequence with $P_n(1) = 1$ for $n = 0, 1, 2, \ldots$. Assume that the recurrence coefficients γ_n, $n = 1, 2, \ldots$, of the monic version satisfy*

$$\gamma_1 \le \tfrac{1}{2}, \qquad \gamma_n \le \tfrac{1}{4}, \qquad n = 2, 3, \ldots.$$

Then, $(P_n)_{n=0}^\infty$ satisfies property (T).

We note that by Wall's comparison theorem [6] any sequence $(\gamma_n)_{n=1}^\infty$ bounded as in Proposition (2.4) is a chain sequence. A result by Nevai can be used to connect property (P) to property (T).

Proposition (2.5). *Let $(P_n(x))_{n=0}^\infty$ be an orthogonal sequence with $\operatorname{supp}\pi = [-1, 1]$ and $P_n(1) = 1$ for $n = 0, 1, 2, \ldots$. Define the linearization coefficients by*

$$P_m(x)P_n(x) = \sum_{k=|m-n|}^{m+n} g(m, n, k)P_k(x)$$

and the connection coefficients by

$$P_n(x) = \sum_{k=0}^{n} a_{n,k}T_k(x).$$

Assume further $c_n \to \frac{1}{2}$ as $n \to \infty$. Then,

$$a_{n,0} = \lim_{m\to\infty} g(m, m, n)\frac{h(m)}{h(n)} \tag{2.1}$$

and

$$a_{n,k} = \lim_{m\to\infty} g(m, m+k, n)\frac{2(h(m)h(m+k))^{1/2}}{h(n)} \quad \text{for } k = 1, \ldots, n. \tag{2.2}$$

Proof. It is shown in Nevai [25, Theorem 13, p. 45] that for any continuous function f on $[-1, 1]$

$$\lim_{m\to\infty} \int_{-1}^{1} f(x)p_m(x)p_{m+n}(x)\, d\pi(x) = \frac{1}{\pi}\int_{-1}^{1} f(x)T_n(x)(1-x^2)^{-1/2}\, dx$$

holds, provided the orthogonalization measure π is an element of $M(0, 1)$ (in the terminology of the quoted work). In our note $\pi \in M(0, 1)$ means exactly $\gamma_n \to \frac{1}{4}$ as $n \to \infty$. Setting $f(x) = P_k(x)$ the assertions (2.1), (2.2) follow. □

Theorem (2.6). *Assume that $(P_n)_{n=0}^\infty$ determines a polynomial hypergroup on $\mathbb{N}_0$, and assume that the defining recurrence coefficients c_n converge to c, $0 < c \le \frac{1}{2}$. Then, $(P_n)_{n=0}^\infty$ satisfies condition* (T).

Proof. The result follows immediately by Proposition (2.5), provided $c = \frac{1}{2}$. For $0 < c < \frac{1}{2}$ set $b = 2\sqrt{c(1-c)}$ and define $Q_n(x) = T_n(x/b)/k_n$, $k_n = T_n(1/b) \ge 1$. The orthogonal polynomial satisfy

$$xQ_n(x) = \frac{bk_{n+1}}{2k_n}Q_{n+1}(x) + \frac{bk_{n-1}}{2k_n}Q_{n-1}(x)$$

and $Q_0(x) = 1$, $Q_1(x) = x$. The recurrence coefficients γ_n of the corresponding monic version are given by

$$\gamma_1 = \frac{b^2}{2}, \qquad \gamma_n = \frac{b^2}{4}, \qquad n = 2, 3, \ldots.$$

Thus, by Proposition (2.4) we have

$$Q_n(x) = \sum_{k=0}^{n} b_{n,k} T_k(x) \tag{1}$$

with $b_{n,k} \geq 0$. The result of Nevai [25, Theorem 13, p. 45] yields also

$$\lim_{m\to\infty} \int_{-1}^{1} P_k(x) p_m(x) p_{m+n}(x)\, d\pi(x) = \frac{k_n}{\pi} \int_{-b}^{b} P_k(x) Q_n(x) (b^2 - x^2)^{-1/2}\, dx.$$

Since $(b^2 - x^2)^{-1/2}\, dx$ restricted to $[-b, b]$ is the orthogonalization measure of the $Q_n(x)$, it follows

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} Q_k(x) \tag{2}$$

with nonnegative $a_{n,k}$'s. (1) and (2) obviously imply (T). □

Remark. The hint to use Nevai's result for Proposition (2.5) came from M. Rösler. The results of Propositions (2.3), (2.4), (2.5) and Theorem (2.6) are also valid for nonsymmetric orthogonalization measures π. The simple modifications to be made follow immediately from the results in [1 and 25]. In the shorter sections below we investigate some classes of orthogonal polynomials whose hypergroup structure does not seem to be revealed yet. But before, we glimpse at the ultraspherical polynomials to see how the results above fit in this well-known class. The usual recurrence formula of ultraspherical polynomials is

$$(n+1)\widetilde{P}_{n+1}(x;\alpha) = 2x(n + \alpha + \tfrac{1}{2})\widetilde{P}_n(x;\alpha) - (2\alpha + n)\widetilde{P}_{n-1}(x;\alpha), \quad n = 1, 2, \ldots, \tag{2.3}$$

and

$$\widetilde{P}_0(x;\alpha) = 1; \qquad \widetilde{P}_1(x;\alpha) = (2\alpha + 1)x, \tag{2.4}$$

see [28, (4.7.17)] (we have a change in the parametrization to $\lambda = \alpha + \frac{1}{2}$). Since

$$\widetilde{P}_n(1;\alpha) = \frac{(2\alpha + 1)_n}{n!} \tag{2.5}$$

see [28, (4.7.3)] and since

$$d_n = \frac{n + 2\alpha}{2n + 2\alpha + 1}, \qquad e_n = 1 - d_n,$$

we have by (1.9)

$$c_n = \frac{n}{2n + 2\alpha + 1}. \tag{2.6}$$

In particular, we see that $\gamma_n = d_n e_{n-1} = c_n(1 - c_{n-1})$ is a chain sequence. Theorem (1.3) applied to the recurrence relation defined by the c_n and $1 -$

c_n implies a polynomial hypergroup $\mathbb{N}_0$. Since $c_n \to \frac{1}{2}$, we have $\operatorname{supp} \pi = [-1, 1] = D_s = D$. That (H) and (T) hold follows by Proposition (2.3) and Theorem (2.6) respectively, whereas

$$h(n) = \frac{(2n+2\alpha+1)(2\alpha+1)_n}{(2\alpha+1)n!}, \tag{2.7}$$

$$\sigma_n = \frac{2^n(\alpha+\frac{1}{2})_n}{(2\alpha+1)_n}, \tag{2.8}$$

$$\rho_n = 2^n \left(\frac{(\alpha+\frac{3}{2})_n(\alpha+\frac{1}{2})_n}{n!(2\alpha+1)_n} \right)^{1/2}, \tag{2.9}$$

holds for all $n \in \mathbb{N}$.

3. Associated ultraspherical polynomials

Fix $\alpha > -\frac{1}{2}$, $\nu \geq 0$. The associated ultraspherical polynomials $\widetilde{P}_n^{(\nu)}(x; \alpha)$ satisfy the recurrence relation

$$\begin{aligned}(n+\nu+1)\widetilde{P}_{n+1}^{(\nu)}(x;\alpha) &= 2x(n+\nu+\alpha+\tfrac{1}{2})\widetilde{P}_n^{(\nu)}(x;\alpha) \\ &\quad - (2\alpha+n+\nu)\widetilde{P}_{n-1}^{(\nu)}(x;\alpha), \qquad n = 1, 2, \dots,\end{aligned} \tag{3.1}$$

and

$$\widetilde{P}_0^{(\nu)}(x;\alpha) = 1, \qquad \widetilde{P}_1^{(\nu)}(x;\alpha) = \frac{2\alpha+2\nu+1}{\nu+1}x. \tag{3.2}$$

Associated ultraspherical polynomials are studied in [36], [5, §3] and for $\alpha = 0$ in [4 and 19]. The corresponding monic polynomials are given by

$$\phi_n^{(\nu)}(x;\alpha) = \frac{(\nu+1)_n}{2^n(\alpha+\nu+\frac{1}{2})_n}\widetilde{P}_n^{(\nu)}(x;\alpha), \tag{3.3}$$

satisfying

$$x\phi_n^{(\nu)}(x;\alpha) = \phi_{n+1}^{(\nu)}(x;\alpha) + \gamma_n\phi_{n-1}^{(\nu)}(x;\alpha), \tag{3.4}$$

where

$$\gamma_n = \frac{(n+\nu+2\alpha)(n+\nu)}{(2n+2\nu+2\alpha+1)(2n+2\nu+2\alpha-1)}, \qquad n = 1, 2, \dots,$$

and

$$\phi_0^{(\nu)}(x;\alpha) = 1, \qquad \phi_1^{(\nu)}(x;\alpha) = x.$$

Setting $\lambda_n = n+\nu+2\alpha$, $\mu_n = n+\nu$ for $n = 0, 1, 2, \dots$, we see that

$$d_n = \frac{\lambda_n}{\lambda_n+\mu_{n+1}} = \frac{n+2\alpha+\nu}{2n+2\alpha+2\nu+1}, \qquad n = 0, 1, 2, \dots,$$

is a parameter sequence for $(\gamma_n)_{n=1}^{\infty}$, provided $d_0 \geq 0$, i.e., $2\alpha+\nu \geq 0$. Proposition (1.6) yields that $(\gamma_n)_{n=1}^{\infty}$ is also a chain sequence, when $2\alpha+\nu < 0$. By (1.13) we have

$$\widetilde{P}_n^{(\nu)}(1;\alpha) = 1 + \sum_{k=1}^{n} \frac{(2\alpha+\nu)_k}{(1+\nu)_k} = \frac{(2\alpha+\nu)_{n+1} - (\nu)_{n+1}}{2\alpha(\nu+1)_n}, \qquad n = 1, 2, \dots, \tag{3.5}$$

where the second equality is formula (7.1.1) of [14, p. 151]. According to (1.9) we set

$$c_n = \frac{\widetilde{P}^{(\nu)}_{n-1}(1;\alpha)(n+2\alpha+\nu)}{\widetilde{P}^{(\nu)}_n(1;\alpha)(2n+2\alpha+2\nu+1)} = \frac{(\nu+n)(2\alpha+\nu)_{n+1}-(n+2\alpha+\nu)(\nu)_{n+1}}{(2n+2\alpha+2\nu+1)[(2\alpha+\nu)_{n+1}-(\nu)_{n+1}]} \tag{3.6}$$

for $n = 1, 2, \ldots$.

For the positivity of the linearization coefficients we observe that $\gamma_n \le \gamma_{n+1}$ is valid for $n = 1, 2, \ldots$, if and only if $\alpha \ge \frac{1}{2}$. Hence, Corollary (1.4) yields a polynomial hypergroup structure on $\mathbb{N}_0$ if $\alpha \ge \frac{1}{2}$. If $-\frac{1}{2} < \alpha \le \frac{1}{2}$ and $2\alpha + \nu \ge 0$, it is easily established that condition (b) of Corollary (1.5) is satisfied. To check the case $2\alpha + \nu < 0$ we use Theorem (1.3), the recurrence relation being defined by the c_n and $1 - c_n$. A tedious calculation shows that $c_n \le c_{n+1}$ is equivalent to

$$(\nu)^2_{n+1}(\nu+n+1)(1-2\alpha) - 2(\nu)_{n+1}(2\alpha+\nu)_{n+1}(n+\alpha+\nu+1)(1-2\alpha)(1+2\alpha) + (2\alpha+\nu)^2_{n+1}(2\alpha+\nu+n+1)(1+2\alpha) \ge 0$$

which obviously holds if $2\alpha + \nu \le 0$. Since $\gamma_n \to \frac{1}{4}$, we have $c_n \to \frac{1}{2}$. Thus $c_n \le \frac{1}{2} \le 1 - c_n$ is valid, too. Now by Theorem (1.3) the c_n determine a polynomial hypergroup even for $2\alpha + \nu \le 0$.

Theorem (3.1). *Fix $\alpha > -\frac{1}{2}$, $\nu \ge 0$. Define c_n by (3.6). Then, by means of*

$$xP^{(\nu)}_n(x;\alpha) = (1-c_n)P^{(\nu)}_{n+1}(x;\alpha) + c_nP^{(\nu)}_{n-1}(x;\alpha),$$
$$P^{(\nu)}_0(x;\alpha) = 1, \qquad P^{(\nu)}_1(x;\alpha) = x$$

a polynomial hypergroup on $\mathbb{N}_0$ is determined. Its dual space may be identified with $D = D_s = [-1, 1]$, bearing the orthogonalization measure $d\pi(x) = f(x)\,dx$ with

$$f(\cos t) = \frac{(\sin t)^{2\alpha}}{|{}_2F_1(\frac{1}{2}-\alpha, \nu; \nu+\alpha+\frac{1}{2}; e^{2it})|^2}, \qquad 0 \le t \le \pi, \tag{3.7}$$

up to a multiplicative constant. The Haar weights are given by

$$h(n) = \frac{(2n+2\alpha+2\nu+1)}{4\alpha^2(2\alpha+2\nu+1)(\nu+1)_n(2\alpha+\nu+1)_n}((2\alpha+\nu)_{n+1}-(\nu)_{n+1})^2. \tag{3.8}$$

Further, $(P^{(\nu)}_n(x, \alpha))^\infty_{n=0}$ satisfies properties (H) *and* (T).

Proof. We have only to note that (3.8) follows from (1.17), whereas the orthogonalization measure is determined in [26], see [5], too. □

The leading coefficients of the associated ultraspherical polynomials are determined by (1.20), resp. (1.21) as

$$\sigma_n = 2^n \frac{\alpha 2^{n+1}(\alpha+\nu+\frac{1}{2})_n}{(2\alpha+1)_{n+1}-(\nu)_{n+1}}, \tag{3.9}$$

resp.

$$\rho_n = 2^n \left(\frac{(\nu+\alpha+\frac{3}{2})_n(\nu+\alpha+\frac{1}{2})_n}{(\nu+1)_n(\nu+2\alpha+1)_n}\right)^{1/2}. \tag{3.10}$$

4. Pollaczek polynomials

Fix $\alpha > -\frac{1}{2}$, $\mu \geq 0$. Define the orthogonal polynomials $\widetilde{P}_n(x; \alpha, \mu)$ by

$$\begin{aligned}(n+1)\widetilde{P}_{n+1}(x; \alpha, \mu) &= 2x(n+\alpha+\mu+\tfrac{1}{2})\widetilde{P}_n(x; \alpha, \mu) \\ &\quad - (n+2\alpha)\widetilde{P}_{n-1}(x; \alpha, \mu), \qquad n = 1, 2, \ldots,\end{aligned} \tag{4.1}$$

and

$$\widetilde{P}_0(x; \alpha, \mu) = 1, \qquad \widetilde{P}_1(x; \alpha, \mu) = (2\alpha + 2\mu + 1)x. \tag{4.2}$$

These orthogonal polynomials—we shall call them Pollaczek polynomials—were first studied by Pollaczek [26] and Szegö [28], see also [6, p. 184]. The corresponding monic polynomials are given by

$$\phi_n(x; \alpha, \mu) = \frac{n!}{2^n(\alpha+\mu+\frac{1}{2})_n}\widetilde{P}_n(x; \alpha, \mu) \tag{4.3}$$

and satisfy

$$x\phi_n(x; \alpha, \mu) = \phi_{n+1}(x; \alpha, \mu) + \gamma_n\phi_{n-1}(x; \alpha, \mu), \tag{4.4}$$

where

$$\gamma_n = \frac{n(n+2\alpha)}{(2n+2\alpha+2\mu+1)(2n+2\alpha+2\mu-1)}, \qquad n = 1, 2, \ldots,$$

and

$$\phi_0(x; \alpha, \mu) = 1, \qquad \phi_1(x; \alpha, \mu) = x.$$

Using the recurrence relation of the Laguerre polynomials $L_n^{(\alpha)}(x)$, see [28, (5.1.10)], we obtain

$$\widetilde{P}_n(1; \alpha, \mu) = L_n^{(2\alpha)}(-2\mu) = (2\alpha+1)_n \sum_{k=0}^{n} \frac{(2\mu)^k}{(2\alpha+1)_k(n-k)!k!}, \tag{4.5}$$

where the second equality is (5.1.6) of [28].

We recognize that $\widetilde{P}_n(1; \alpha, \mu)$ is positive for each $n \in \mathbb{N}$. Therefore, $(\gamma_n)_{n=1}^{\infty}$ is a chain sequence. In fact, set

$$d_n = \frac{n+2\alpha}{(2n+2\alpha+2\mu+1)}, \qquad e_n = \frac{n+1}{(2n+2\alpha+2\mu+1)},$$

and

(4.6)

$$\begin{aligned}c_n &= \frac{\widetilde{P}_{n-1}(1; \alpha, \mu)}{\widetilde{P}_n(1; \alpha, \mu)} d_n \\ &= \frac{n}{(2n+2\alpha+2\mu+1)} \left(\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(2\mu)^k}{(2\alpha+1)_k}\right) \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(2\mu)^k}{(2\alpha+1)_k}\right)^{-1} \\ &\qquad \text{for } n = 1, 2, \ldots.\end{aligned}$$

By (1.10) we have $1 - c_n = \widetilde{P}_{n+1}(1; \alpha, \mu)e_n/\widetilde{P}_n(1; \alpha, \mu)$. Hence, $0 < c_n < 1$ holds for $n = 1, 2, \ldots$ and $(1 - c_{n-1})c_n = d_n e_{n-1} = \gamma_n$ is valid.

If $\alpha \geq \frac{1}{2}$, we obtain $\gamma_n \geq \gamma_{n+1}$ for $n = 1, 2, \ldots$. Further, we have $d_n \leq e_n$, $d_n \leq d_{n+1}$ and $d_n + e_n \leq d_{n+1} + e_{n+1}$ for $n = 1, 2, \ldots$, provided $-\frac{1}{2} < \alpha \leq$

$\frac{1}{2}$. But $e_0 \leq d_1 + e_1$ is not satisfied for some $\mu \geq 0$ if $-\frac{1}{2} < \alpha < 0$. A transformation of random walk polynomials enables us to cope with the case $-\frac{1}{2} < \alpha < 0$ (see §7). In fact, in §7 we derive that a polynomial hypergroup is determined, if $-\frac{1}{2} < \alpha < 0$ and $0 \leq \mu < \alpha + \frac{1}{2}$.

Theorem 4.1. *Fix* $\alpha \geq 0$, $\mu \geq 0$, *or, let* $-\frac{1}{2} < \alpha < 0$, $0 \leq \mu < \alpha + \frac{1}{2}$. *Define* c_n *by* (4.6). *Then, by means of*

$$xP_n(x; \alpha, \mu) = (1 - c_n)P_{n+1}(x; \alpha, \mu) + c_n P_{n-1}(x; \alpha, \mu),$$
$$P_0(x; \alpha, \mu) = 1, \qquad P_1(x; \alpha, \mu) = x$$

a polynomial hypergroup on $\mathbb{N}_0$ *is determined. Its dual space may be identified with* $D = D_s = [-1, 1]$ *bearing the orthogonalization measure* $d\pi(x) = f(x)\,dx$ *with*

(4.7)
$$f(\cos t) = (\sin t)^{2\alpha}|\Gamma(\alpha + \tfrac{1}{2} + i\mu\cot(t))|^2 \exp((2t - \pi)\mu\cot(t)), \qquad 0 \leq t \leq \pi,$$

up to a multiplicative constant. The Haar weights are given by

$$h(n) = \frac{(2n + 2\alpha + 2\mu + 1)(2\alpha + 1)_n}{(2\alpha + 2\mu + 1)n!} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(2\mu)^k}{(2\alpha + 1)_k}\right)^2. \tag{4.8}$$

Also, $(P_n(x; \alpha, \mu))_{n=0}^{\infty}$ *satisfies properties* (H) *and* (T).

Proof. The orthogonalization measure is determined in [26] (see also [6]). All other statements follow from the result in §§1 and 2. □

The leading coefficients of the Pollaczek polynomials are

$$\sigma_n = \frac{2^n(\alpha + \mu + \frac{1}{2})_n}{(2\alpha + 1)_n} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(2\mu)^k}{(2\alpha + 1)_k}\right)^{-1} \tag{4.9}$$

resp.

$$\rho_n = 2^n \left(\frac{(\alpha + \mu + \frac{3}{2})_n(\alpha + \mu + \frac{1}{2})_n}{n!(2\alpha + 1)_n}\right)^{1/2}. \tag{4.10}$$

We do point out that the orthogonalization function $f(x)$ of the Pollaczek polynomials in (4.7) does not satisfy Szegö's condition on $[-1, 1]$. A weight function $f(x)$ on $[-1, 1]$ is said to satisfy Szegö's condition if $\ln f(x)(1 - x^2)^{-1/2}$ is integrable in $[-1, 1]$.

5. Associated Pollaczek polynomials

Fix $\alpha > -\frac{1}{2}$, $\mu \geq 0$, $\nu \geq 0$. Define the assciated Pollaczek polynomials $\widetilde{P}_n^{(\nu)}(x; \alpha, \mu)$ by

$$\begin{aligned}(n + \nu + 1)\widetilde{P}_{n+1}^{(\nu)}(x; \alpha, \mu) &= 2x(n + \nu + \alpha + \mu + \tfrac{1}{2})\widetilde{P}_n^{(\nu)}(x; \alpha, \mu) \\ &\quad - (n + \nu + 2\alpha)\widetilde{P}_{n-1}^{(\nu)}(x; \alpha, \mu), \qquad n = 1, 2, \ldots.\end{aligned} \tag{5.1}$$

$$\widetilde{P}_0^{(\nu)}(x; \alpha, \mu) = 1, \qquad \widetilde{P}_1^{(\nu)}(x; \alpha, \mu) = \frac{2\alpha + 2\nu + 2\mu + 1}{\nu + 1}x. \tag{5.2}$$

These polynomials are the associated ones of those considered in §4. They were first studied by Pollaczek [26], see also [6, p. 185]. The corresponding monic polynomials are given by

(5.3) $$\phi_n^{(\nu)}(x;\alpha,\mu)=\frac{(\nu+1)_n}{2^n(\alpha+\nu+\mu+\frac{1}{2})_n}\widetilde{P}_n^{(\nu)}(x;\alpha,\mu)$$

and satisfy

(5.4) $$\phi_n^{(\nu)}(x;\alpha,\mu)=\phi_{n+1}^{(\nu)}(x;\alpha,\mu)+\gamma_n\phi_{n-1}^{(\nu)}(x;\alpha,\mu),$$

where

$$\gamma_n=\frac{(n+\nu+2\alpha)(n+\nu)}{(2n+2\nu+2\alpha+2\mu+1)(2n+2\nu+2\alpha+2\mu-1)},\qquad n=1,2,\dots,$$

and

$$\phi_0^{(\nu)}(x;\alpha,\mu)=1,\qquad \phi_n^{(\nu)}(x;\alpha,\mu)=x.$$

The value of $\widetilde{P}_n^{(\nu)}(1;\alpha,\mu)$ is given by $L_n^{(2\alpha)}(-2\mu;\nu)$, where $L_n^{(\alpha)}(x;\nu)$ denotes the associated Laguerre polynomials studied in [3 and 15]. In fact, the associated Laguerre polynomials $L_n^{(\alpha)}(x;\nu)$ are generated by

$$\begin{aligned}(n+\nu+1)L_{n+1}^{(\alpha)}(x;\nu)&=(-x+2n+2\nu+\alpha+1)L_n^{(\alpha)}(x;\nu)\\&\quad-(n+\nu+\alpha)L_{n-1}^{(\alpha)}(x;\nu),\end{aligned}$$

$$L_0^{(\alpha)}(x;\nu)=1,\qquad L_1^{(\alpha)}(x;\nu)=(-x+2\nu+\alpha+1)/(\nu+1).$$

Hence, we see that

(5.5) $$\widetilde{P}_n^{(\nu)}(1,;\alpha,\mu)=L_n^{(2\alpha)}(-2\mu;\nu).$$

Furthermore, the leading coefficient of $L_n^{(\alpha)}(x;\nu)$ is $(-1)^n/(\nu+1)_n$ as is easily derived from their recurrence relation. (This follows also from (3.4) in [36] or (2.8) in [3], where an explicit representation of $L_n^{(\alpha)}(x;\nu)$ is established.)

Since the zeros of the associated Laguerre polynomials lie in $]0,\infty[$ positivity of $L_n^{(2\alpha)}(-2\mu;\nu)$ is shown. Setting

$$d_n=\frac{n+\nu+2\alpha}{2n+2\nu+2\alpha+2\mu+1},\qquad e_n=\frac{n+\nu+1}{2n+2\nu+2\alpha+2\mu+1}$$

and

(5.6) $$c_n=\frac{L_{n-1}^{(2\alpha)}(-2\mu;\nu)}{L_n^{(2\alpha)}(-2\mu;\nu)}d_n\quad\text{for } n=1,2,\dots$$

with $1-c_n=L_{n+1}^{(2\alpha)}(-2\mu;\nu)e_n/L_n^{(2\alpha)}(-2\mu;\nu)$ we see that $0<c_n<1$ and $(1-c_{n-1})c_n=\gamma_n$ is valid.

For $-\frac{1}{2}<\alpha\le\frac{1}{2}$ we have $d_n\le e_n$, $d_n\le d_{n+1}$ and $d_n+e_n\le d_{n+1}+e_{n+1}$ for $n=1,2,\dots$. Further, $e_0\le d_1+e_1$ if $0\le\alpha\le\frac{1}{2}$. For $\alpha\ge\frac{1}{2}$ the γ_n are nondecreasing. Thus, Theorem (1.3) resp. Corollary (1.4) yields

Theorem (5.1). *Fix $\alpha \geq 0$, $\mu \geq 0$, $\nu \geq 0$. Define c_n by* (5.6). *Then, by means of*

$$xP_n^{(\nu)}(x;\alpha,\mu) = (1-c_n)P_{n+1}^{(\nu)}(x;\alpha,\mu) + c_n P_{n-1}^{(\nu)}(x;\alpha,\mu),$$
$$P_0^{(\nu)}(x;\alpha,\mu) = 1, \qquad P_1^{(\nu)}(x;\alpha,\mu) = x,$$

a polynomial hypergroup on $\mathbb{N}_0$ is determined. Its dual space may be identified with $D = D_s = [-1,1]$, bearing the orthogonalization measure $d\pi(x) = f(x)\,dx$ with

$$f(\cos t) = (\sin t)^{2\alpha}|\Gamma(\alpha + \tfrac{1}{2} + \nu + i\mu\cot(t))|^2 \exp((2t-\pi)\mu\cot(t)) \cdot |{}_2F_1(\tfrac{1}{2} - \alpha + i\cot(t), \nu; \alpha + \tfrac{1}{2} + \nu + i\mu\cot(t); e^{2it})|^{-2}, \quad 0 \leq t \leq \pi, \tag{5.7}$$

up to a multiplicative constant. The Haar weights are given by

$$h(n) = \frac{(2n+2\nu+2\alpha+2\mu+1)(\nu+1)_n}{(2\alpha+2\nu+2\mu+1)(\nu+2\alpha+1)_n}(L_n^{(2\alpha)}(-2\mu;\nu))^2 \tag{5.8}$$

Further, $(P_n^{(\nu)}(x;\alpha,\mu))_{n=0}^{\infty}$ satisfies properties (H) *and* (T).

Proof. The orthogonalization measure is determined in [26]. All other statements follow as before. □

Finally, the leading coefficients of the associated Pollaczek polynomials are

$$\sigma_n = \frac{2^n(\alpha+\nu+\mu+\frac{1}{2})_n}{(\nu+1)_n L_n^{(2\alpha)}(-2\mu;\nu)}, \tag{5.9}$$

respectively

$$\rho_n = 2^n\left(\frac{(\alpha+\nu+\mu+\frac{3}{2})_n(\alpha+\nu+\mu+\frac{1}{2})_n}{(\nu+1)_n(\nu+2\alpha+1)_n}\right)^{1/2}. \tag{5.10}$$

We point out that in [18, 3(e)] we were only able to reveal the polynomial hypergroup structure for rather restricted values of α, ν and μ. The condition $e_0 \leq d_1 + c_1$ is satisfied even for $-\frac{1}{2} < \alpha < 0$, if ν and μ obey a certain inequality. We conjecture that Theorem (5.1) holds also in the situation where $-\frac{1}{2} < \alpha < 0$, $\nu \geq 0$ and $0 \leq \mu < \alpha + \frac{1}{2}$.

6. Orthogonal polynomials with constant monic recursion formula

Fix $0 < \gamma \leq \frac{1}{4}$, $0 < \alpha \leq 1/2\gamma$. Consider monic orthogonal polynomials determined by

$$x\phi_n(x;\gamma,\alpha) = \phi_{n+1}(x;\gamma,\alpha) + \gamma_n\phi_{n-1}(x;\gamma,\alpha), \quad n = 1,2,\ldots, \tag{6.1}$$
$$\phi_0(x;\gamma,\alpha) = 1, \qquad \phi_1(x;\gamma,\alpha) = x,$$

where

$$\gamma_n = \gamma \quad \text{for } n = 2,3,\ldots \quad \text{and} \quad \gamma_1 = \alpha\gamma. \tag{6.3}$$

There are some interesting special cases among these polynomials $\phi_n(x;\frac{1}{4},2)$ are the monic Tchebicheff polynomials of the first kind, and $\phi_n(x;\frac{1}{4},1)$ are the monic Tchebicheff polynomials of the second kind. Moreover, $\phi_n(x; \frac{1}{4}, \alpha)$ define the Geronimus polynomials studied in [13]. Their hypergroup structure we have studied in [18, 3(g)(i)] (in the notation of [18] we have $\alpha = 4/a$). Also, polynomials connected with homogeneous trees are among them, compare [18, 3(d)] (in the notation of [18] set $\gamma = (a-1)/a^2$, $\alpha = a/(a-1)$). In the following we consider $0 < \gamma < \frac{1}{4}$. For shortness denote $\omega = \sqrt{1-4\gamma}$. Set

$$d_0 = 1 - \alpha(1+\omega)/2, \quad e_0 = 1 - d_0 = \alpha(1-\omega)/2$$

and

$$d_n = \tfrac{1}{2}(1-\omega), \quad e_n = 1 - d_n = \tfrac{1}{2}(1+\omega) \quad \text{for } n = 1, 2, \ldots$$

and

$$\widetilde{P}_n(x; \gamma, \alpha) = \frac{1}{e_1^n \alpha}\phi_n(x; \gamma, \alpha), \qquad n = 1, 2, \ldots.$$

The polynomials $\widetilde{P}_n(x; \gamma, \alpha)$ satisfy the recurrence relation (1.1), (1.2) with the above d_n, e_n. Using (1.11) a direct calculation gives for $n = 1, 2, \ldots$

(6.4)

$$\begin{aligned}\phi_n(1; \gamma, \alpha) &= \frac{\alpha}{2^n}(1+\omega)^n \widetilde{P}_n(1; \gamma, \alpha)\\ &= [(2-2\alpha)((1+\omega)^n - (1-\omega)^n) + \alpha((1+\omega)^{n+1} - (1-\omega)^{n+1})]/(2^{n+1}\omega).\end{aligned}$$

By Wall's comparison theorem the sequence $(\gamma_n)_{n=1}^{\infty}$ defined in (6.3) is a chain sequence. The minimal parameter sequence for $(\gamma_n)_{n=1}^{\infty}$ is given by

$$c_1 = \alpha\gamma \tag{6.5}$$

and

$$c_n = \frac{2\gamma[(2-2\alpha)((1+\omega)^{n-1} - (1-\omega)^{n-1}) + \alpha((1+\omega)^n - (1-\omega)^n)}{(2-2\alpha)((1+\omega)^n - (1-\omega)^n) + \alpha((1+\omega)^{n+1} - (1-\omega)^{n+1})} \quad \text{for } n = 2, 3, \ldots. \tag{6.6}$$

(For $\gamma = \frac{1}{4}$ we derive

$$c_1 = \frac{\alpha}{4}, \; c_n = \frac{\alpha + (2-\alpha)(n-1)}{2(\alpha + (2-\alpha)n)},$$

compare [18, 3(g)(i)].) Obviously, $d_n \le e_n$, $d_n \le d_{n+1}$ and $e_n + d_n = 1$ for $n = 1, 2, \ldots$. Hence, for using Theorem (1.3) we only have to check whether $e_0 \le 1$ is valid. But $e_0 \le 1$ is equivalent to the condition $\alpha \le 2$ and $\gamma \ge (\alpha-1)/\alpha^2$. Therefore, the c_n of (6.5), (6.6) determine a polynomial hypergroup, provided $0 < \gamma \le \frac{1}{4}$, $0 < \alpha \le 2$ and $\gamma \ge (\alpha-1)\alpha^{-2}$. The region described by these (α, γ) is bounded by $\gamma = \frac{1}{4}$ at the top and by $\gamma = (\alpha-1)\alpha^{-2}$ at the right side. The curve at the right side corresponds to the polynomials connected with homogeneous trees, the curve at the top to the Geronimus polynomials.

Theorem (6.1). *Fix γ, α such that $0 < \gamma < \frac{1}{4}$, $0 < \alpha \le 2$, $\gamma \ge (\alpha-1)\alpha^{-2}$. Define c_n by (6.5), (6.6). Then by means of*

$$xP_n(x; \gamma, \alpha) = (1-c_n)P_{n+1}(x; \gamma, \alpha) + c_n P_{n-1}(x; \gamma, \alpha),$$
$$P_0(x; \gamma, \alpha) = 1, \qquad P_1(x; \gamma, \alpha) = x,$$

a polynomial hypergroup on $\mathbb{N}_0$ *is determined. We have* $\operatorname{supp}\pi = [-2\sqrt{\gamma}, 2\sqrt{\gamma}]$, $D_s = [-1, 1]$ *and* $D = \{z \in \mathbb{C}: |z - 2\sqrt{\gamma}| + |z + 2\sqrt{\gamma}| \leq 2\}$. *The orthogonalization measure is given by* $d\pi(x) = f(x)\,dx$, *where*

$$f(x) = \frac{\sqrt{4\gamma - x^2}}{\alpha^2\gamma - (\alpha - 1)x^2} \quad \textit{for } -2\sqrt{\gamma} \leq x \leq 2\sqrt{\gamma}. \tag{6.7}$$

The Haar weights are

$$h(1) = \frac{1}{\alpha\gamma} \tag{6.8}$$

and

$$h(n) = \frac{[(2-2\alpha)((1+\omega)^n - (1-\omega)^n) + \alpha((1+\omega)^{n+1} - (1-\omega)^{n+1})]^2}{\alpha\gamma^n 4^{n+1}(1-4\gamma)} \tag{6.9}$$

for $n = 2, 3, \ldots,$ where $\omega = \sqrt{1 - 4\gamma}$.

Property (H) does not hold, whereas (T) is valid.

Proof. A simple change of variables yields the orthogonalization measure. In fact, set

$$Q_n(x; \gamma, \alpha) = \frac{1}{(2\sqrt{\gamma})^n}\phi_n(2\gamma x; \gamma, \alpha).$$

Then the $Q_n(x; \gamma, \alpha)$ satisfy the recurrence relation of the $\phi_n(x; \frac{1}{4}, \alpha)$. Hence, $Q_n(x; \gamma, \alpha) = \phi_n(x; \frac{1}{4}, \alpha)$. The orthogonalization measure of the Geronimus polynomials $\phi_n(x; \frac{1}{4}, \alpha)$ is known, see [6, p. 205] or [18, 3(g)(i)], and we obtain that the support of the weight function $f(x)$ corresponding to the $\phi_n(x; \gamma, \alpha)$ is $[-2\sqrt{\gamma}, 2\sqrt{\gamma}]$ and

$$f(x) = \frac{\sqrt{4\gamma - x^2}}{\alpha^2\gamma - (\alpha - 1)x^2} \quad \text{for } x \in [-2\sqrt{\gamma}, 2\sqrt{\gamma}].$$

The other assertions concerning D_s, D and the properties (H) and (T) follow from §2. □

The leading coefficients of these polynomials are

$$\sigma_n = \frac{2^{n+1}\omega}{(2-2\alpha)((1+\omega)^n - (1-\omega)^n) + \alpha((1+\omega)^{n+1} - (1-\omega)^{n+1})} \tag{6.10}$$

resp.

$$\rho_n = \frac{1}{\sqrt{\alpha\gamma^n}} \quad \text{for } n = 1, 2, \ldots \text{ (if } \gamma \neq \tfrac{1}{4}). \tag{6.11}$$

Finally, we mention that

$$\phi_n\left(1; \frac{1}{4}, \alpha\right) = \frac{\alpha}{2^n}\left(1 + \frac{2-\alpha}{\alpha}n\right).$$

Hence, for $\gamma = \frac{1}{4}$ the Haar weights are

$$h(1) = \frac{4}{\alpha} \quad \text{and} \quad h(n) = \alpha\left(1 + \frac{2-\alpha}{\alpha}n\right)^2 \quad \text{for } n = 2, 3, \ldots, \tag{6.12}$$

whereas their leading coefficients are $\sigma_n = 1/\phi_n(1; \frac{1}{4}, \alpha)$ resp. $\rho_n = 2^n/\alpha$.

A further reference for orthogonal polynomials with a constant recursion formula is [8], where these polynomials are applied to harmonic analysis on certain discrete groups.

7. Random walk polynomials

In order to define the recurrence relation of random walk polynomials we use the normalization $P_n(1) = 1$. Fix $a \geq 1$, $b \geq 0$. Determine the random walk polynomials $P_n(x; a, b)$ as

$$\begin{aligned} xP_n(x; a, b) &= \frac{an+b}{(a+1)n+b} P_{n+1}(x; a, b) \\ &\quad + \frac{n}{(a+1)n+b} P_{n-1}(x; a; b), \qquad n = 1, 2, \dots, \end{aligned} \tag{7.1}$$

$$P_0(x; a, b) = 1, \qquad P_1(x; a, b) = x. \tag{7.2}$$

Have in mind that for $a = 1$ the polynomials $P_n(x; 1, b)$ are the ultraspherical polynomials $P_n(x; (b-1)/2)$. For $b = 0$ the polynomials $P_n(x; a, 0)$ coincide with the polynomials connected with homogeneous trees, compare [18, 3(d)]. Random walk polynomials are studied in [2, §6]. They are associated to random walks which in turn describe the states of a linear growth birth and death process.

Setting $d_n = c_n = n/(a+1)n + b$ and $e_n = 1 - c_n$ it is easy to verify that the assumptions of Theorem 1.3 hold, if $a \geq 1$, $b \geq 0$. Therefore, the $(c_n)_{n=0}^{\infty}$ determine a polynomial hypergroup structure on $\mathbb{N}_0$. In order to find the dual space and the orthogonalization measure we carry out the following transformation, compare [2]. Let $a > 1$, $b > 0$. Setting

$$Q_n(x; a, b) = \frac{(\sqrt{a})^n (b/a)_n}{n!} P_n\left(\frac{2\sqrt{a}}{a+1}x; a, b\right), \tag{7.3}$$

formulae (7.1) and (7.2) become

$$\begin{aligned} (n+1)Q_{n+1}(x; a, b) &= 2x\left(n + \frac{b}{a+1}\right) Q_n(x; a, b) \\ &\quad - \left(n + \frac{b}{a} - 1\right) Q_{n-1}(x; a, b) \quad \text{for } n = 1, 2, \dots \end{aligned} \tag{7.4}$$

and

$$Q_0(x; a, b) = 1, \qquad Q_1(x; a, b) = \frac{2b}{a+1}x. \tag{7.5}$$

This recurrence relation resembles that of the Pollaczek polynomials. In fact, with

$$\alpha = \frac{1}{2}\left(\frac{b}{a} - 1\right) \quad \text{and} \quad \mu = b\left(\frac{1}{a+1} - \frac{1}{2a}\right)$$

we obtain

$$Q_n(x; a, b) = \widetilde{P}_n(x; \alpha, \mu) \quad \text{for } n = 0, 1, 2, \dots, \tag{7.6}$$

where the $\widetilde{P}_n(x; \alpha, \mu)$ are the Pollaczek polynomials defined by (4.1) and (4.2).

Theorem (7.1). *Fix* $a > 1$, $b > 0$. *Then, by means of* (7.1) *and* (7.2) *a polynomial hypergroup on* $\mathbb{N}_0$ *is determined. We have*

$$\operatorname{supp}\pi = \left[-2\frac{\sqrt{a}}{a+1}, 2\frac{\sqrt{a}}{a+1}\right], \qquad D_s = [1, 1]$$

and

$$D = \left\{z \in \mathbb{C}\colon \left|z - 2\frac{\sqrt{a}}{a+1}\right| + \left|z + 2\frac{\sqrt{a}}{a+1}\right| \le 2\right\}.$$

The orthogonalization measure is given by $d\pi(x) = f(x)\,dx$, *where* (*up to a multiplicative constant*)

$$f(x) = s(x)^{(b/a-1)}\left|\Gamma\left(\frac{1}{2}\frac{b}{a} + i\frac{b(a-1)}{2a}\frac{x}{s(x)}\right)\right|^2 \cdot \exp\left((2\arccos x - \pi)\frac{b(a-1)}{2a}\frac{x}{s(x)}\right), \tag{7.7}$$

with

$$s(x) = \sqrt{4a - x^2(a+1)^2} \quad \textit{for} \quad -2\frac{\sqrt{a}}{a+1} \le x \le 2\frac{\sqrt{a}}{a+1}.$$

The Haar weights are given by

$$h(n) = \frac{a^n(\frac{b}{a})_n((a+1)n+b)}{bn!}. \tag{7.8}$$

Further, $(P_n(x; a, b))_{n=0}^{\infty}$ *satisfies property* (T), *but not* (H).

Proof. The orthogonalization measure can be established from the corresponding one in §4. The other statements follow from §§1 and 2. □

The leading coefficients of the random walk polynomials are

$$\sigma_n = \left(\frac{a+1}{a}\right)^n \left(\frac{b}{a+1}\right)_n \Big/ \left(\frac{b}{a}\right)_n, \tag{7.9}$$

resp.

$$\rho_n = \left(\frac{a+1}{a}\right)^n \left(\frac{b}{a+1}\right)_n \left(\frac{a^n((a+1)n+b)}{bn!(b/a)_n}\right)^{1/2} \quad \text{for } n \in \mathbb{N}. \tag{7.10}$$

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